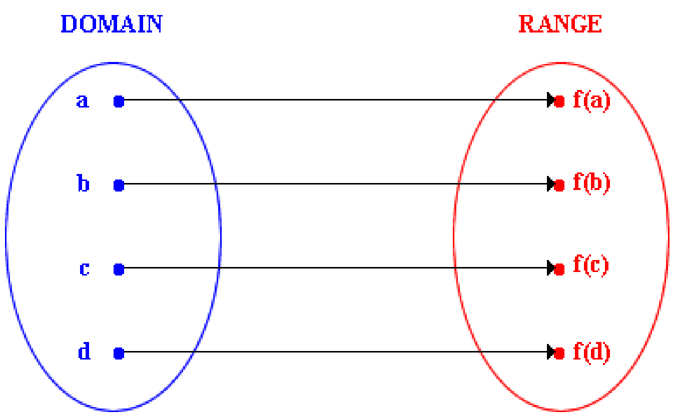
Refresher Course in Calculus, Probability, and Statistics

Day 1: Calculus

Function

➲ Definition:

- A function of a real variable *x* with domain D is a rule that assigns a unique real number to each number *x* in D.
- Given a function *f*, the set of numbers x at which ***f(x)*** is defined is called the **domain** of *f*. The set of values *f(x)* obtained as *x* varies in the domain is called the **range** of *f*.

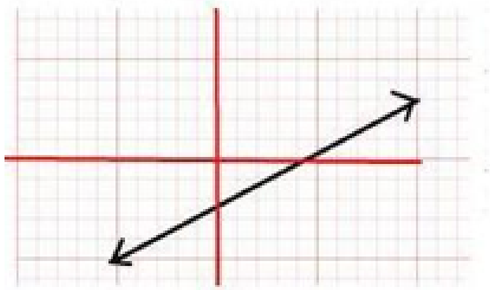


- Example: a supply function y = a + bp, where y is output, p is price.

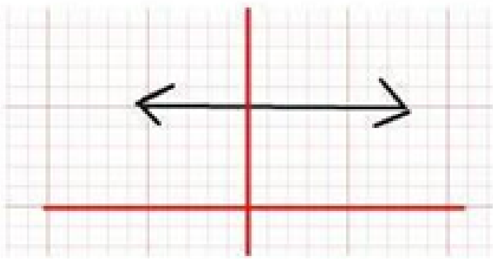
Function Forms:

➲ Examples:

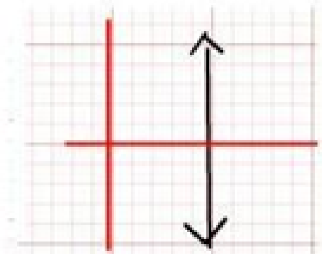
- Linear function: $y = ax + b$
- Quadratic function: $y = ax^2 + bx + c$
- Natural exponential function: $y = e^x$
- Nature logarithmic function: $y = \ln x$



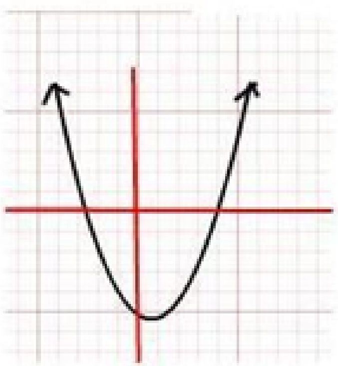
Linear



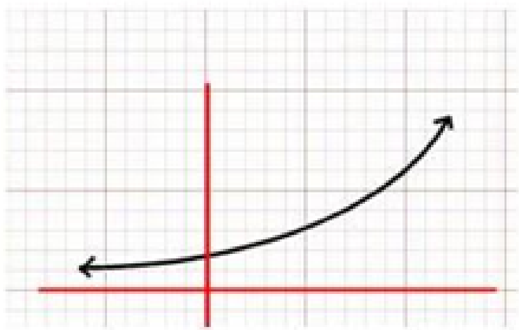
Constant



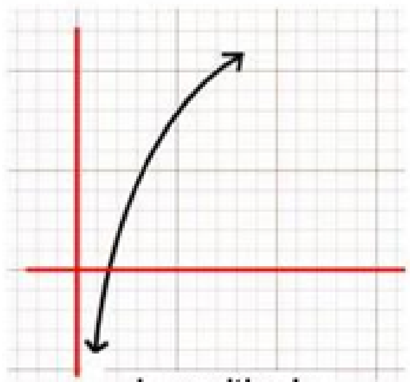
Non function



Quadratic



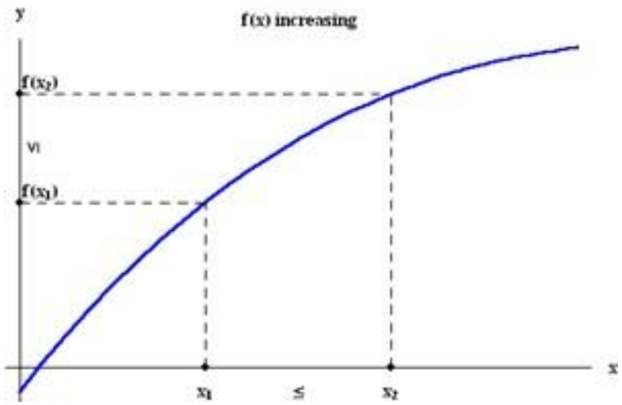
Exponential



Logarithmic

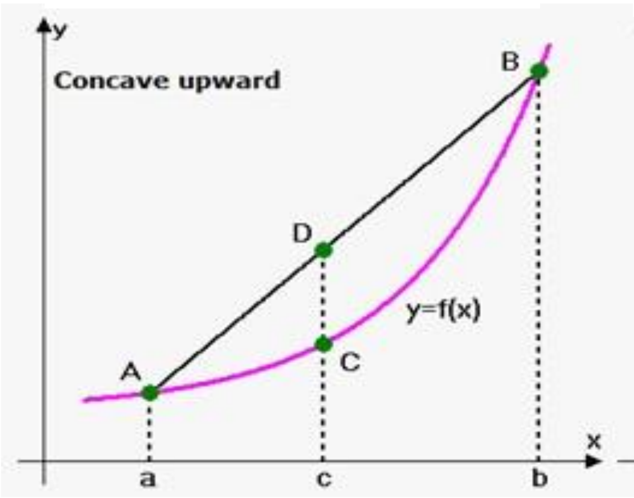
Geometric Properties of a Function:

- Increasing or decreasing

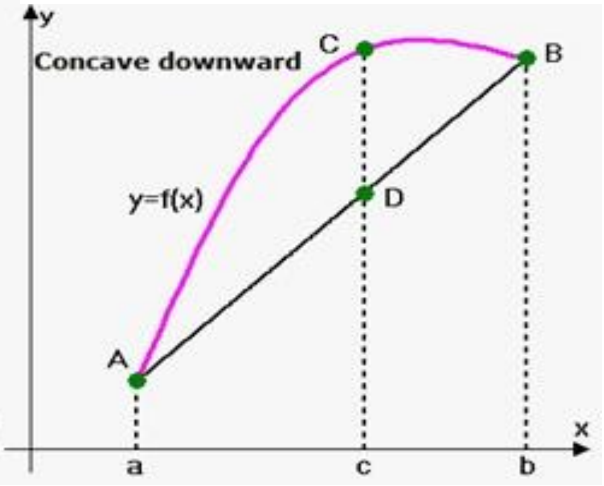


- Concave or convex

Convex



Concave



❖ Derivative

Introduction

- Differential calculus is the study of the rate at which quantities change.
  - The primary objects of study in differential calculus are the **derivatives** of a **primitive** function.
  - The process of finding a derivative is called differentiation.
- Integral calculus can be regarded as the "inverse" of differentiation.
  - The process of reconstructing a function from its derivative is called integration.
  - 2 kinds of integrals:
    - indefinite integral → a function
    - definite integral → a number

❖ **Focus on differential calculus and know how to get a derivative function**

- References:
  - [CAR] chap. 4; [CIA] chap. 6-8, 10, 13; [DOW] chap. 3-5, 7, 8, 14-16, 18; [SBL] chap. 2.4-2.7, 3.5, 4, 5.5, 14; [SH1] chap. 6-9; [SH2] chap. 5 & 6
  - *Paul's Online Math Notes* [URL: http://tutorial.math.lamar.edu/] Calculus I, Calculus II (Integration techniques)

Difference Quotient

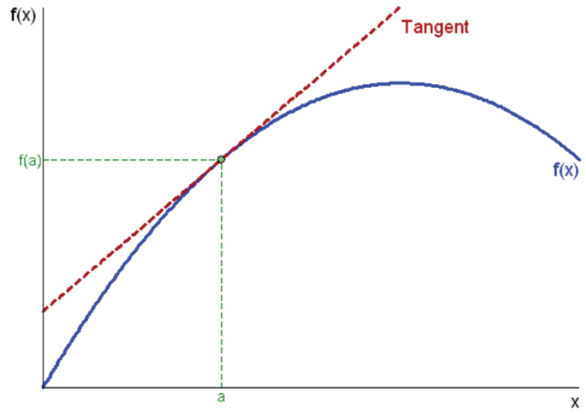
- A change in a variable x is denoted: $\Delta x = x_1 - x_0$
- Corresponding change of function: $y = f(x)$
  - $\Delta y = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$

Difference Quotient: Change in y per unit of change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- expresses the average change in y for $\Delta x = 1$ on $[x_0, x_1]$

Derivative : Formal Definitions



Derivative of function $y = f(x)$

➲ At point x_0:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- $f'(x_0)$ is the *instantaneous/marginal* rate of change
- $f' \geq 0 \ (> 0) \Leftrightarrow f$ increasing (strictly increasing)
- $f' \leq 0 \ (< 0) \Leftrightarrow f$ decreasing (strictly decreasing)

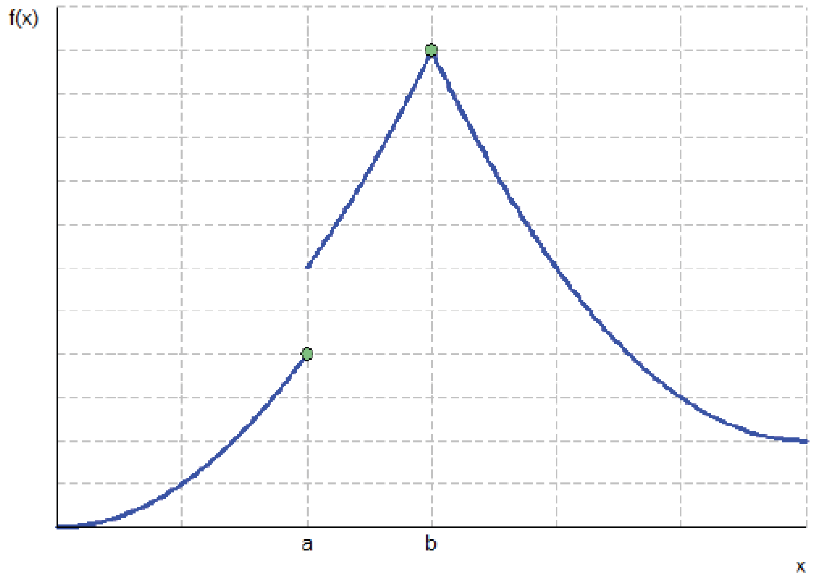
➲ At any point x:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- $f'(x)$ is a function describing the *instantaneous* rate of change of $y = f(x)$ for all values of x

Differentiability and Continuity

- Function is said **differentiable** at a point if its derivative exists at that point.
- To be differentiable at a point, a function must:
  - be continuous AND
  - have a unique tangent



Notation of the derivative

- Derivative of $y = f(x)$ may be written in several *equivalent* ways:

$$f'(x) \qquad y' \qquad f_x(x) \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{df(x)}{dx} \qquad \frac{d}{dx}f(x)$$

- Leibniz's notation (with *d*s)
  - preferred in economics
  - emphasizes that $f(x)$ differeniated with respect to (wrt) x
  - can prevent confusion for functions of several variables $f(x_1, \dots, x_n)$ and higher-order derivatives

Isaac Newton (1642-1727)

Gottfried Wilhelm Leibniz (1646-1716)

Basic Rules for Differentiation

Function $f(x)$	Derivative $\frac{df(x)}{dx}$
α	0
αx	α
x^{α}	$\alpha x^{\alpha-1}$
α^{x}	$\alpha^{x}\ln(\alpha)$
e^{x}	e^{x}
$\ln(x)$	$\frac{1}{x}$

Simple Examples

- Compute $f'(x)$ when $f(x) = x^3$:
  - Using definition (let: $\Delta x = h$)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} \\
&= \lim_{h\to 0} 3x^2+3xh+h^2 \\
&= 3x^2
\end{aligned}
$$

  - Using power function rule $f(x) = x^\alpha \Longrightarrow \frac{df(x)}{dx} = \alpha x^{\alpha-1}$

$$\frac{dx^3}{dx} = 3x^2$$

- Compute $f'(x)$ when $f(x) = \frac{1}{x}$ (Note: $\frac{1}{x} = x^{-1}$)

$$\frac{df(x)}{dx} = (-1)x^{-2} = -\frac{1}{x^2}$$

Differentiation of Sums, Products, and Quotients

Function	Derivative
$h(x)$	$\dfrac{dh(x)}{dx}$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$

Chain Rule for Differentiation

- Composite function (or *function of function*): function in which y is function of u and u in turn is function of x

$$y = f(u) \qquad \text{and} \qquad u = g(x) \qquad \Longrightarrow \qquad y = f[g(x)]$$

- Derivative of y wrt x:

$$y' = f'(u) \cdot g'(x) = f'[g(x)] \cdot g'(x)$$

or: $$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{dg}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Example: $f(x) = (1 - x^3)^5, \qquad (g(x) = 1 - x^3, f(x) = g(x)^5)$

$$f'(x) = 5 \cdot (1 - x^3)^4 \cdot (-3x^2) = -15x^2 \cdot (1 - x^3)^4$$

Higher Order Derivatives

- ➲ $f'(x)$: **first derivative** of f
- ➲ $f''(x)$: **second-order derivative** of f

$$f''(x) = \frac{d}{dx}\left[\frac{df(x)}{dx}\right] = \frac{d^2 f(x)}{dx^2}$$

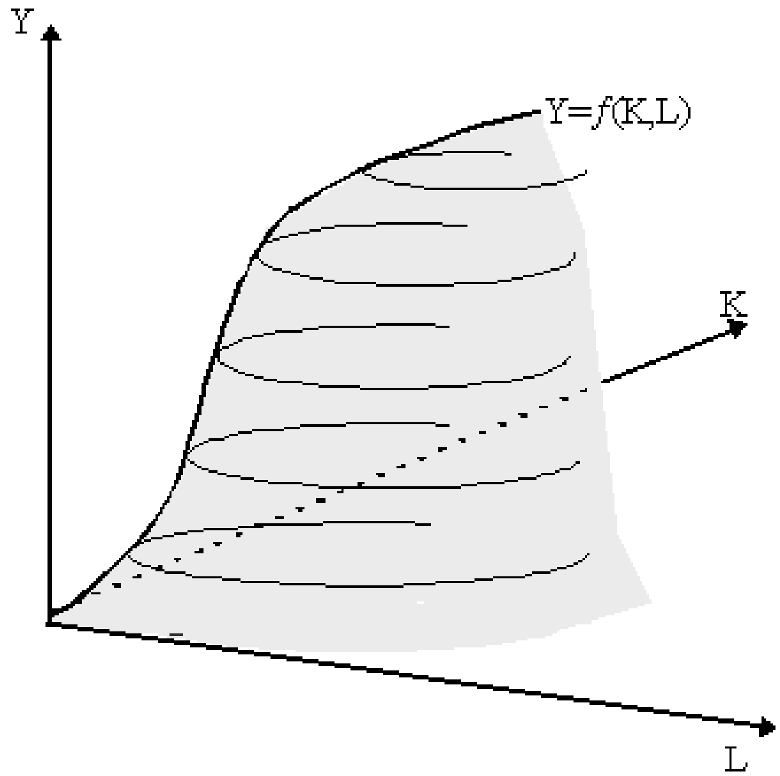
  - o $f''(x) > 0 \iff f$ **convex** (at x)
  - o $f''(x) < 0 \iff f$ **concave** (at x)

- ➲ $f'''(x) = \frac{d^3 f(x)}{dx^3}$: **third-order derivative** of f
- ➲ $f''''(x) = \frac{d^4 f(x)}{dx^4}$: **fourth-order derivative** of f
- ➲ …

Multi-variable Functions

➲ Functions of several variables

- Example: a production function y = AK^aL^b
- measures rate of change of y wrt x_1 while x_2 is held constant (Note: "*held constant*" is used equivalently to "*all other things being equal*" or "*ceteris paribus*" in economics.)



Multivariable Functions and partial derivative

- For functions of several variables $y = f(x_1, x_2)$, **(first-order) partial derivative**:
  - $\frac{\partial y}{\partial x_1}$: (first-oder) partial derivative of $f(x_1, x_2)$ with respect to (wrt) x_1
  - ∂ (read: «(partial) dee» or «partial») has the same function as *d* in Leibniz notation
  - measures rate of change of y wrt x_1 while x_2 is held constant (Note: "*held constant*" is used equivalently to "*all other things being equal*" *or* "*ceteris paribus*" in economics.)

- Notations: $\frac{\partial y}{\partial x_1}, \frac{\partial f}{\partial x_1}, f_1(x_1, x_2), f_{x_1}(x_1, x_2), y_1$

- Partial differentiation wrt one of the independent variables follows the same rules as ordinary differentiation *with other variable(s)/argument(s) treated as constants*

Second-Order Partial Derivatives

- Second-order **direct** partial derivatives of $y = f(x_1, x_2)$
  - wrt x_1: $f_{11} = f_{x_1x_1} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}y = \frac{\partial^2 y}{\partial x_1^2}$
  - wrt x_2: $f_{22} = f_{x_2x_2} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}y = \frac{\partial^2 y}{\partial x_2^2}$

- **Cross (or mixed)** partial derivatives:

$$f_{12} = f_{x_1x_2} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}y = \frac{\partial^2 y}{\partial x_1 x_2}$$
$$f_{21} = f_{x_2x_1} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}y = \frac{\partial^2 y}{\partial x_2 x_1}$$

  - If both cross partial derivatives are continuous , they are identical (i.e. $f_{12} = f_{21}$ [Young's theorem])

Exponential Functions

- Exponential function:

exponent

$$f(x) = a^x \qquad\qquad a > 0$$

base

- Properties:
  - Same base:

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{x \cdot y}$$

  - Same exponent:

$$a^x \cdot b^x = (ab)^x$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

- Derivative: $f'(x) = a^x \ln(a)$

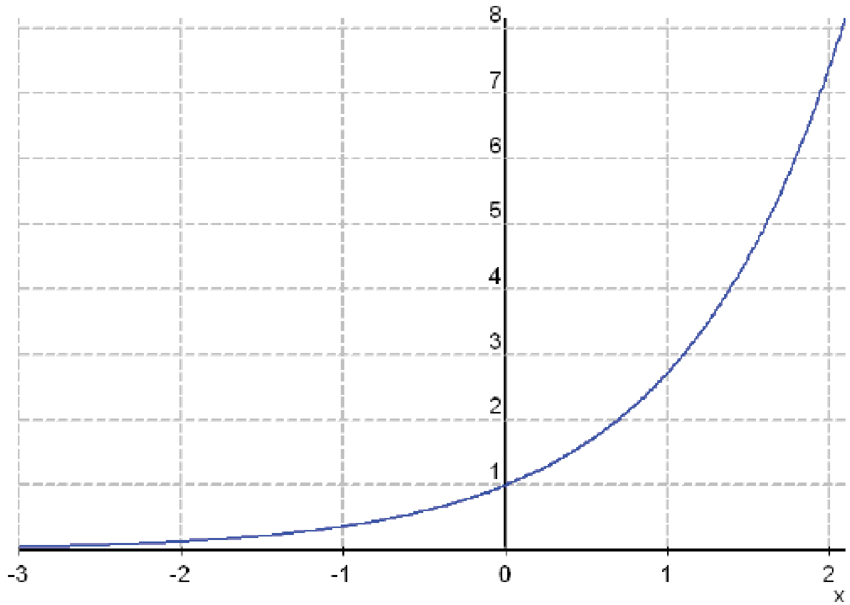
Natural Exponential Function

- Natural exponential function:

$$f(x) = e^x = exp(x)$$

with $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828\ldots$ (Euler's number)

- Derivatives: $f'(x) = e^x, f''(x) = e^x, \ldots$



Logarithmic Functions

- Logarithmic function with base a:

exponent

$$f(x) = \log_a x \qquad\qquad a > 0, a \neq 1$$

base

Definition: $\log_a x$ is the exponent to which a must be raised to obtain x

$$y = a^x \iff x = \log_a y$$

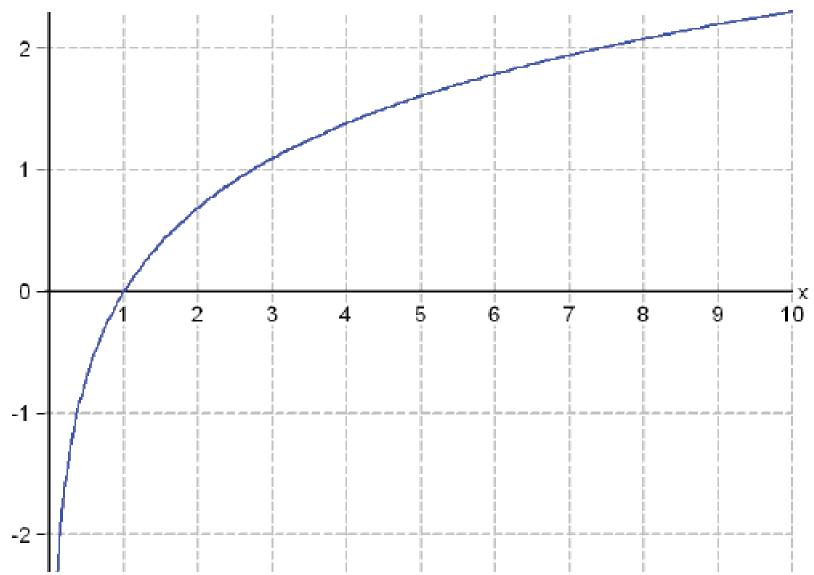
- Properties:
  - $a^{\log_a x} = \log_a a^x = x$
  - $\log_a(x \cdot y) = \log_a x + \log_a y$
  - $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - $\log_a x^n = n \cdot \log_a x$ (also applies to: $\log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} = \frac{1}{n} \cdot \log_a x$)
  - $\log_a x = (\log_a c)(\log_c x) \iff \log_c x = \frac{\log_a x}{\log_a c}$
- Derivative: $f'(x) = \frac{1}{x \ln a}$

Natural Logarithmic Function

- Natural logarithmic function:

$$f(x) = \log_e x = \ln x$$

- Inverse function of e^x: $e^{\ln x} = \ln(e^x) = x$
- Derivatives: $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, \ldots$



Logarithmic Differentiation

- Logarithmic differentiation: $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$ iff $f(x)$ is differentiable
  - Demonstration: suppose that
    $y = \ln f(x)$ and $f(x)$ is differentiable
  - Recall, this is a composite function $y = g[f(x)]$:
    $g(u) = \ln u$ and $u = f(x)$
  - Differentiate both functions wrt u and x, respectively:
    $g' = \frac{1}{u}$ and $u' = f'(x)$
  - Apply chain rule: $y' = g'(u) \cdot f'(x)$:
    $$y' = g' \cdot u' = \frac{1}{u} f'(x) = \frac{f'(x)}{u}$$
  - Recall, that $u = f(x)$ and substitute
    $$y' = \frac{f'(x)}{u} = \frac{f'(x)}{f(x)}$$
- Logarithmic differentiation: use this property to differentiate $f(x)$
  $$f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}[\ln f(x)] f(x)$$

Logarithmic Differentiation (continued)

Example ❶:

➲ $f(x) = x^x$

- Taking the logarithm:
  $\ln f(x) = \ln x^x = x \ln x$

- Differentiate both sides wrt x (using logarithmic differentiation on the lhs):
  $$\frac{f'(x)}{f(x)} = 1 \cdot \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

- Multiply both sides by $f(x)$:
  $f'(x) = (\ln x + 1)x^x$

Logarithmic Differentiation (continued)

Example ❷:

➲ $f(x) = [A(x)]^\alpha [B(x)]^\beta$

- Taking the logarithm:
  $\ln f(x) = \ln\left([A(x)]^\alpha [B(x)]^\beta\right) = \alpha \ln A(x) + \beta \ln B(x)$

- Differentiate both sides wrt x (using logarithmic differentiation on the lhs): :
  $$\frac{f'(x)}{f(x)} = \alpha \frac{1}{A(x)} A'(x) + \beta \frac{1}{B(x)} B'(x) = \alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)}$$

- Multiply both sides by $f(x)$:
  $$f'(x) = \left(\alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)}\right) [A(x)]^\alpha [B(x)]^\beta$$

Elasticities

- **Elasticity** of $y = f(x)$ with respect to x
  - Ratio between *relative* change in y and *relative* change in x
  - dimensionless measure of association

$$\varepsilon_{yx} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx} \cdot \frac{x}{y}$$

  - $f(x)$ is $\begin{Bmatrix} elastic \\ of\ unit\ elasticity \\ inelastic \end{Bmatrix}$ at a point when $|\varepsilon_{yx}| \begin{Bmatrix} > 1 \\ = 1 \\ < 1 \end{Bmatrix}$

- Generally: $\varepsilon_{yx} = \frac{dy}{dx} \cdot \frac{x}{y} = \frac{d \ln y}{d \ln x}$
  - Because: $d \ln y = \frac{dy}{y}$ and $d \ln x = \frac{dx}{x}$

- Partial elasticities for functions of multiple variables $y = f(x_1, \dots, x_n)$ are similarly defined:

$$\varepsilon_{yx_1} = \frac{\partial y}{\partial x_1} \cdot \frac{x_1}{y}, \dots, \varepsilon_{yx_n} = \frac{\partial y}{\partial x_n} \cdot \frac{x_n}{y}$$

Elasticities (continued)

Example:

➲ Find elasticity of Y with respect to K for $Y = AK^{\alpha}L^{\beta}$

Definition of elasticity: $\boldsymbol{\varepsilon_{YK} = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y}}$	**Definition of elasticity:** $\boldsymbol{\varepsilon_{YK} = \frac{\partial \ln Y}{\partial \ln K}}$
o $\frac{\partial Y}{\partial K}$ is the partial derivative of Y wrt. K: $\frac{\partial Y}{\partial K} = \alpha AK^{(\alpha-1)}L^{\beta}$ o $\frac{K}{Y}$ is simply the ratio of K and Y: $\frac{K}{Y} = \frac{K}{AK^{\alpha}L^{\beta}} = \frac{1}{AK^{\alpha-1}L^{\beta}}$ o Combining the two terms: $\varepsilon_{YK} = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \alpha AK^{(\alpha-1)}L^{\beta} \cdot \frac{1}{AK^{\alpha-1}L^{\beta}} = \alpha$	o Log transformation: $\ln Y = \ln\left(AK^{\alpha}L^{\beta}\right) = \ln A + \ln K^{\alpha} + \ln L^{\beta} = \ln A + \alpha \ln K + \beta \ln L$ o Taking the partial derivative of $\ln Y$ wrt. $\ln K$: $\frac{\partial \ln Y}{\partial \ln K} = \alpha$

Derivatives (further information and exercises)

➲ More comprehensive discussion of differentiation (on Paul Dawkin's website)

- One variable derivatives:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcI/DerivativeIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Problems/CalcI/DerivativeIntro.aspx

- Partial derivatives:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivsIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivsIntro.aspx

- Applications:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcI/DerivAppsIntro.aspx and http://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivAppsIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Problems/CalcI/DerivAppsIntro.aspx and http://tutorial.math.lamar.edu/Problems/CalcIII/PartialDerivAppsIntro.aspx

Indefinite Integral

Suppose we don't know primitive function H

- All we know is its derivative: $H'(t) = \frac{dH}{dt} = t^{-\frac{1}{2}}$
- We can use the differentiation rule: $f(x) = x^{\alpha} \Longrightarrow f'(x) = \alpha x^{\alpha-1}$
- And reverse it: $f'(x) = x^{\alpha} \Longrightarrow f(x) = \frac{1}{\alpha+1} x^{\alpha+1}$

Possible solution for H

- $H(t) = 2t^{\frac{1}{2}}$
- But what about $H(t) = 2t^{\frac{1}{2}} + 99$ or $\mathrm{H}(t) = 2t^{\frac{1}{2}} + 27000$?
- Because all constants disappear, the indefinite integral:

$$H(t) = 2t^{\frac{1}{2}} + c$$

  is a group of functions, where c is an arbitrary constant
- To explicitly solve $H(t) = 2t^{\frac{1}{2}} + c$ we need additional information
  - *Initial* or *boundary condition* (what was the state of the $H(t)$ at a certain value of t)
    - E.g. $H(0) = 33$ → $H(0) = 33 = 2 * 0^{\frac{1}{2}} + c \iff c = 33$
    - $H(t) = 2t^{\frac{1}{2}} + 33$

Indefinite Integral

- Definition of **indefinite integral**:

$$\int f(x)dx = F(x) + c \qquad \text{when } F'(x) = f(x)$$

- Basic Vocabulary:
  - Symbol $\int$: **integral sign**
  - Function $f(x)$: **integrand**
  - x: **variable of integration** (indicated by dx)
  - c: **constant of integration**

- We say *indefinite* because the technique of integration is applied without any numbers, borders, or limits
  - Result is usually a function

Common integrals

Function	Anti-derivative
$\int x^{\alpha}\, dx$	$\frac{1}{\alpha+1} x^{\alpha+1} + c \quad \text{if } \alpha \neq -1$
$\int \alpha\, dx = \int \alpha\, x^{0}\, dx$	$\alpha x + c$
$\int dx = \int 1\, x^{0}\, dx$	$x + c$
$\int \frac{1}{x} dx$	$\ln\|x\| + c$
$\int \frac{1}{\alpha x + \beta} dx$	$\frac{1}{\alpha} \ln\|\alpha x + \beta\| + c$
$\int e^{x}\, dx$	$e^{x} + c$
$\int e^{\alpha x + b}\, dx$	$\frac{1}{\alpha} e^{\alpha x + b} + c \quad \text{if } \alpha \neq 0$
$\int \alpha^{x}\, dx$	$\frac{1}{\ln \alpha} \alpha^{x} + c \quad \text{if } \alpha > 0 \text{ and } \alpha \neq 1$

Indefinite Integral: properties

- Multiplicative constants

$$\int \alpha f(x)dx = \alpha \int f(x)dx$$

- Multiplicative constant -1

$$\int -f(x)dx = -\int f(x)dx$$

- Sums and differences

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$

Indefinite Integral: examples

➲ Example ❶:

Evaluate the integral:

$$\int x^4 - \frac{3}{2x} + e^{2x} - 9\,dx \quad = \int x^4 dx - \frac{3}{2}\int \frac{1}{x} dx + \int e^{2x} dx - \int 9dx$$

$$= \frac{1}{5}x^5 - \frac{3}{2}\ln|x| + \frac{1}{2}e^{2x} - 9x + C$$

➲ Example ❷:

Evaluate the integral:

$$\int x^4 - \frac{3}{2x} dx \; + e^{2x} - 9 \quad = \int x^4 dx - \frac{3}{2}\int \frac{1}{x} dx + e^{2x} - 9$$

$$= \frac{1}{5}x^5 - \frac{3}{2}\ln|x| + e^{2x} - 9 + C$$

Definite Integral

- Definite integrals involve integrating function between two points (called "limits")
- Definition of **definite integral**:

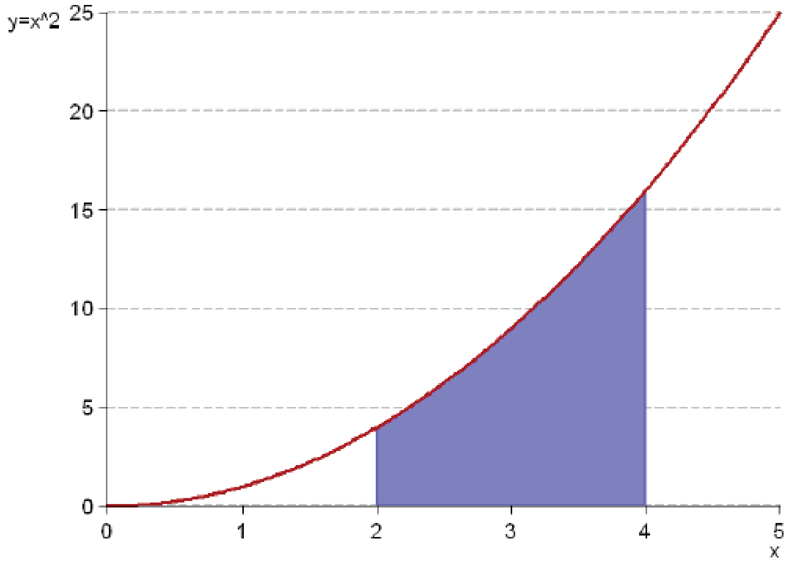
$$\int_a^b f(x)dx = \Big|_a^b F(x) = F(x)\Big|_a^b = F(b) - F(a)$$

  - where F is any function satisfying $F'(x) = f(x)$ for all $x \in [a, b]$
- (geometric) Interpretation
  - If $f(x) > 0$ over $[a, b]$, $\int_a^b f(x)dx$ is the area between the graph of $f(x)$, the x-axis and the lines $x = a$ and $x = b$
  - If $f(x) < 0$ over $[a, b]$, $-\int_a^b f(x)dx$ is the area between the graph of $f(x)$, the x-axis and the lines $x = a$ and $x = b$

Definite Integral: geometric interpretation

Example:

➲ Find area between x-axis and $f(x) = x^2$ over [2, 4] (shaded area in figure below)



$$\int_2^4 x^2 dx = \frac{1}{3}x^3\Big|_2^4 = \frac{1}{3}4^3 - \frac{1}{3}2^3 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3} \approx 18.66$$

Definite integral: Properties

- $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- $$\int_a^a f(x)dx = 0$$

- $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (a \le c \le d)$$

- $$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

- $$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Definite Integral: Present Value of a Future Cash Flow

➲ Net present value = current value of a future reward, payment, …

$$\underbrace{A}_{\text{Net present value}} = \underbrace{V}_{\text{nominal value}} e^{-\overbrace{r}^{\text{discount rate}}\underbrace{t}_{\text{time to reception}}}$$

➲ What about a flow/stream of future payments?

- Sum of all discounted payments at all periods
- $\Pi = \int_0^{\tau} Ve^{-rt}\,dt = V\int_0^{\tau} e^{-rt}\,dt = V * -\frac{1}{r}e^{-rt}\Big|_0^{\tau} = -\frac{V}{r}e^{-r\tau} + \frac{V}{r} = \frac{V}{r}(1 - e^{-r\tau})$
  - Assume: $r = 0.06$, $\tau = 5$ and $V = \$3000$
  - Present value of future cash flow: $\Pi = \frac{\$3000}{0.06}(1 - e^{-0.06*5}) \approx \$12959$

Integral (further information and exercises)

➲ More comprehensive discussion of integration techniques (on Paul Dawkin's website)

- Basics and integration by substitution:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcI/IntegralsIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Problems/CalcI/IntegralsIntro.aspx

- More techniques:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcII/IntTechIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Problems/CalcII/IntTechIntro.aspx

- Multiple Integrals:
  - Explanation: http://tutorial.math.lamar.edu/Classes/CalcIII/MultipleIntegralsIntro.aspx
  - Exercises: http://tutorial.math.lamar.edu/Problems/CalcIII/MultipleIntegralsIntro.aspx