Refresher Course in Calculus,Probability, and Statistics

Day 2: Optimization

Introduction

- Optimization: the process of finding the relative maximum or minimum of a function
- Optimization problems are certainly the most frequent in economics and finance:
  - Maximize profit or minimize cost (producer)
  - Maximize utility (consumer)
  - Maximize return or minimize variance (investor)
  - Minimize sum of squared error when estimating a regression line
  - …
- Often in economics and finance, optimization problems
  - imply constraints $\Longrightarrow$ Constrained optimization
- References:

  [CAR] chap. 5; [CIA] chap. 9, 11, 12; [DOW] chap. 4-6, 20-21 [SBL] chap. 17-19; [SH1] chap. 8, 13-14; [SH2] chap. 3, 8-10

Optimization

- ➲ requires primarily a (relevant) **objective function**
  - o Functional relationship between the object of optimization and **choice variables**:

$$\text{object of optimization} \longrightarrow y = \underbrace{f(\overset{\substack{\text{choice variable(s)}\\ \downarrow}}{x})}_{\text{objective function}}$$

  - o ususally assumed to be (twice) continuously differentiable
- ➲ Example: Maximize profit π
  - o Profit is the difference between total revenue (R) and total costs (C): $\pi = R - C$
  - o Both revenue and costs are a function of the quantity produced (= sold): Q
    - $R = R(Q)$
    - $C = C(Q)$
  - o Objective function of the profit maximization problem is:

$$\pi(Q) = R(Q) - C(Q)$$

  - o the only choice variable is quantity (Q)

Unconstrained Optimization

First-order condition:

- To find (local/global) **extrema** of a function, take first derivative and set it equal to zero:

$$f'(x) = 0$$

- *Necessary* condition known as **first-order condition (FOC)**
  - *Necessary* but not *sufficient*.

- Identifies all points at which the function is neither increasing nor decreasing, but at a plateau.

- All points identified this way are **candidates** (or **critical points**) for a *possible* maximum or minimum

Unconstrained Optimization (continued)

Second-order condition

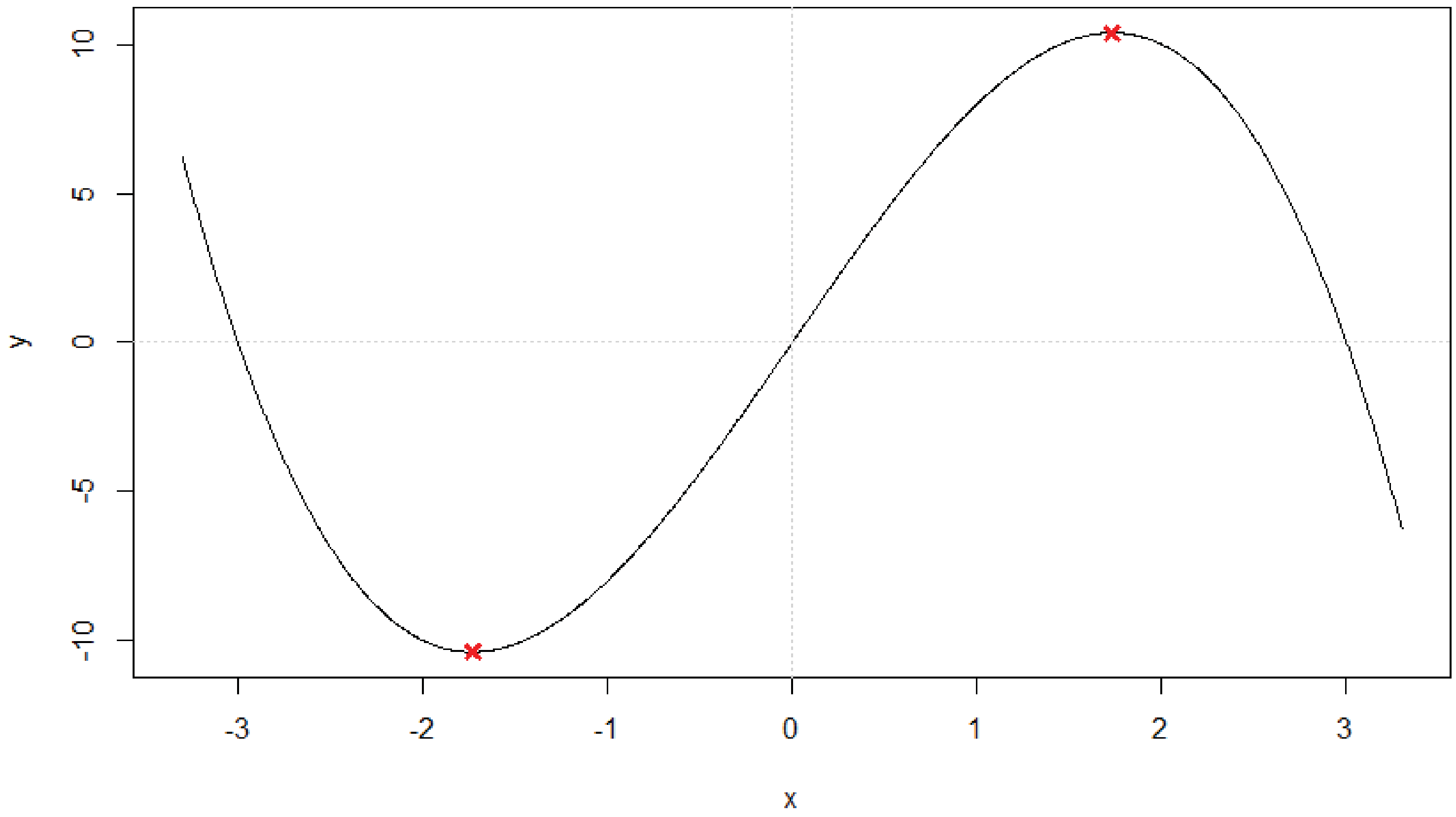
- Once critical points are identified by FOC (i.e. $f'(x) = 0$ for $x = a$):
  - Take second derivative of $f(x)$
  - Evaluate it at each critical point

If…	then function is…	and a is a
$f''(a) < 0$	concave at a	relative maximum
$f''(a) > 0$	convex at a	relative minimum
$f''(a) = 0$	test is inconclusive	

- Assuming necessary FOC met, **second-order condition (SOC)** is a *sufficient* condition to qualify an extremum

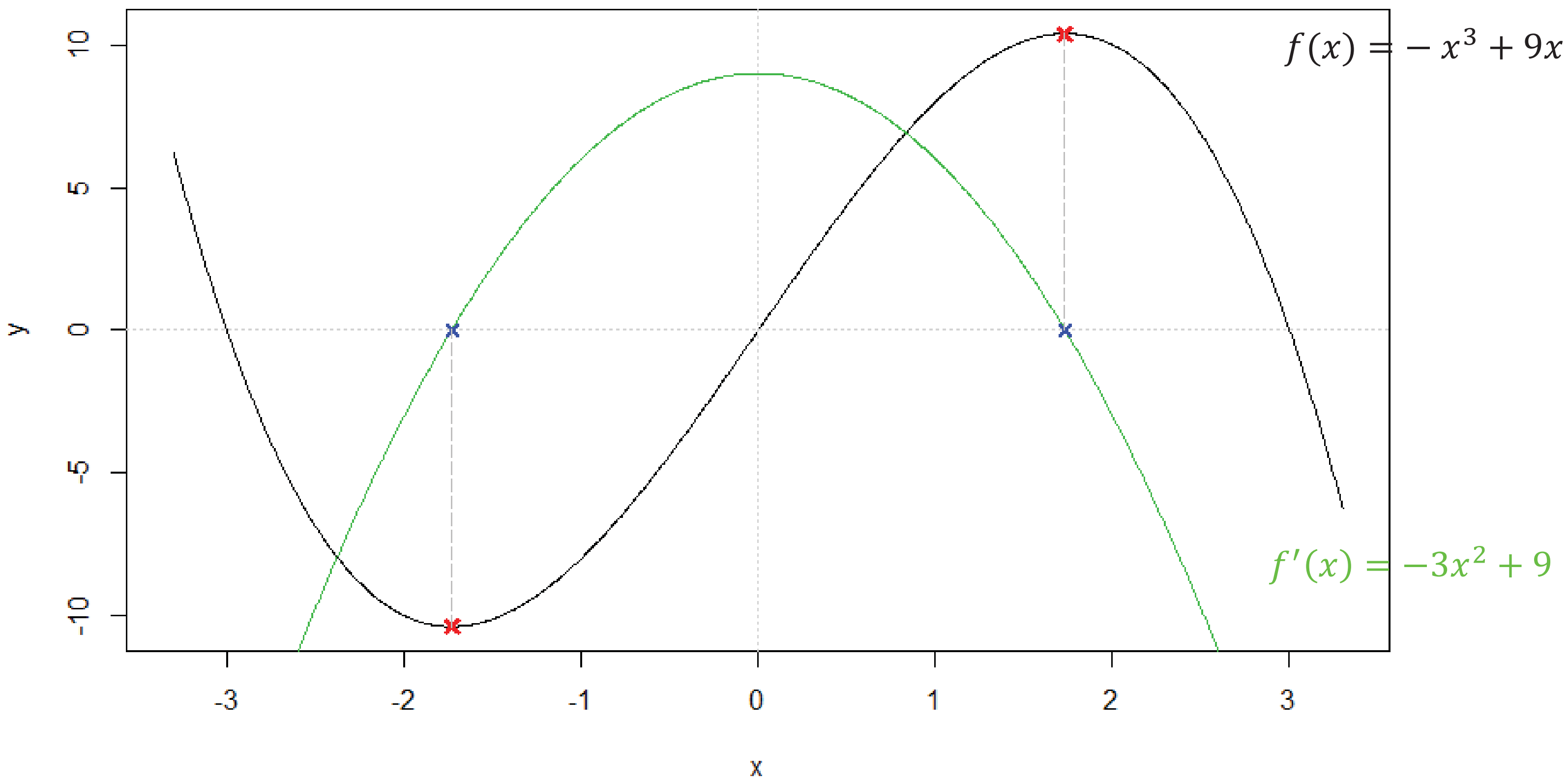
Unconstrained Optimization (continued)

Example: Find extreme points, maximum(s) and minimum(s) of $f(x) = -x^3 + 9x$



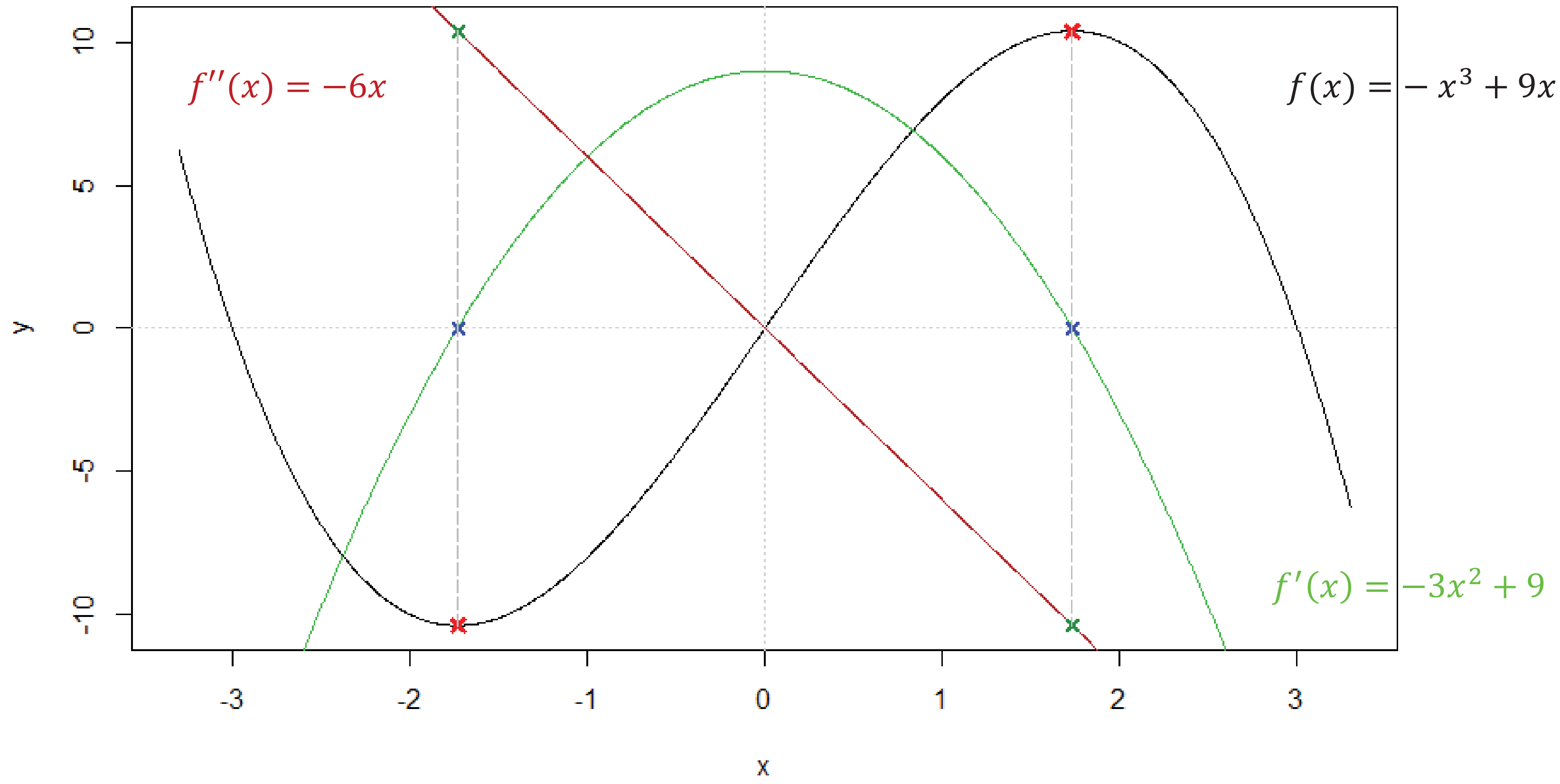
Unconstrained Optimization (continued)

- FOC: $f'(x) = 0 = -3x^2 + 9$
  - roots: $x_{1,2} = \pm\sqrt{3}$
  - Extreme points: $(x_1^*, y_1^*) = \left(\sqrt{3}, 6\sqrt{3}\right)$ and $(x_2^*, y_2^*) = \left(-\sqrt{3}, -6\sqrt{3}\right)$



Unconstrained Optimization (continued)

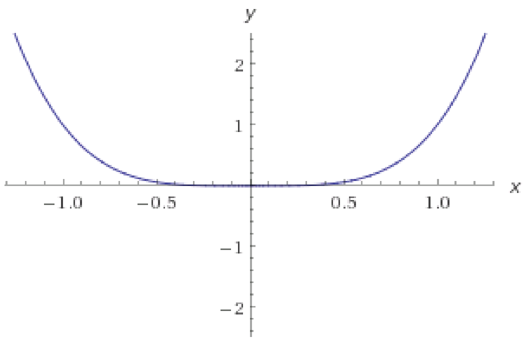
- SOC: $f''(x) = -6x$
  - At $x = x_1^* = \sqrt{3}$: $f''(\sqrt{3}) = -6\sqrt{3} < 0$: $(x_1^*, y_1^*) = (\sqrt{3}, 6\sqrt{3})$ is a maximum
  - At $x = x_2^* = -\sqrt{3}$: $f''(-\sqrt{3}) = 6\sqrt{3} > 0$: $(x_2^*, y_2^*) = (\sqrt{3}, 6\sqrt{3})$ is a minimum



Unconstrained Optimization (continued)

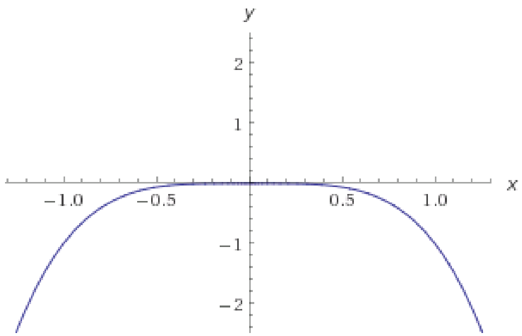
Possible cases where $f'(a) = f''(a) = 0$ (here with $a = 0$)

$y = x^4$



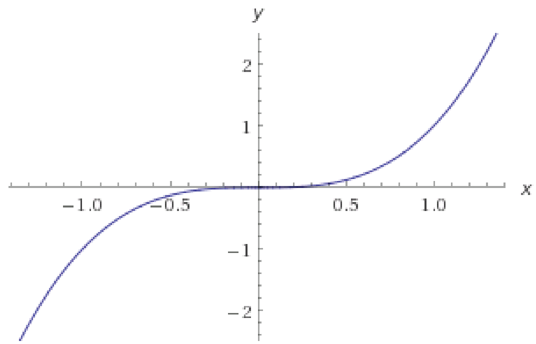
Minimum

$y = -x^4$



Maximum

$y = x^3$



Inflection point

Unconstrained Optimization (continued)

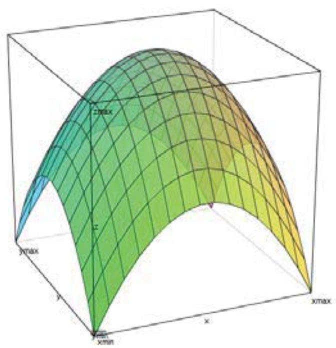
Succesive derivative test

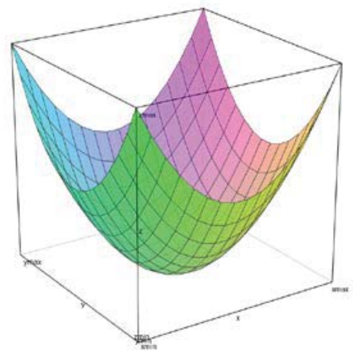
- If $f''(a) = 0$, SOC inconclusive
- Succesive derivative test is helpful:
- Evaluate higher-order derivatives at critical point
- If:
  - 1st nonzero higher-order derivative is an odd-numbered derivative (i.e. 3rd, 5th, etc.):
    - ✓ Inflection point, i.e. a point at which function changes from being convex ($f''(a) > 0$) to being concave ($f''(a) < 0$) or vice versa
  - 1st nonzero higher-order derivative is an even-numbered derivative (i.e. 4th, 6th, etc.) and if the value of this derivative is:
    - ✓ negative $\Longrightarrow$ Maximum
    - ✓ positive $\Longrightarrow$ Minimum

Multivariable Functions (continued)

➲ Searching for extrema of multivariable function $z = f(x, y)$:

Condition(s)	Maximum	Minimum
First-order necessary	$\frac{\partial f(x,y)}{\partial x} = f_x = 0$ and $\frac{\partial f(x,y)}{\partial y} = f_y = 0$	
Second-order necessary	$\frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx} < 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy} < 0$	$\frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx} > 0$ and $\frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy} > 0$
	and	
Second-order sufficient	$f_{xx} \times f_{yy} > (f_{xy})^2 \left(f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} \right)$	



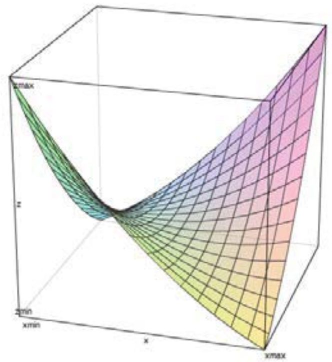


Multivariable Functions (continued)

- Searching for extrema of multivariable function $z = f(x, y)$:
- If:
  - o $f_{xx} \times f_{yy} < (f_{xy})^2$ & f_{xx} and f_{yy} same signs

    → **inflection point**

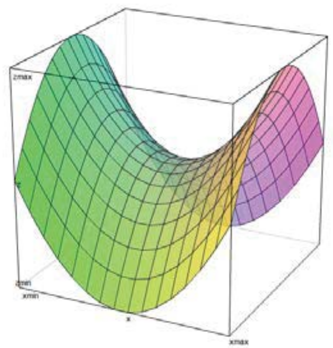
    → e.g. $z = x^2 + y^2 + 3xy$



  - o $f_{xx} \times f_{yy} < (f_{xy})^2$ & f_{xx} and f_{yy} opposite signs

    → **saddle point**

    → e.g., $z = x^2 - y^2$



  - o $f_{xx} \times f_{yy} = (f_{xy})^2$: test is inconclusive

    → e.g., $z = x^3 + y^3$

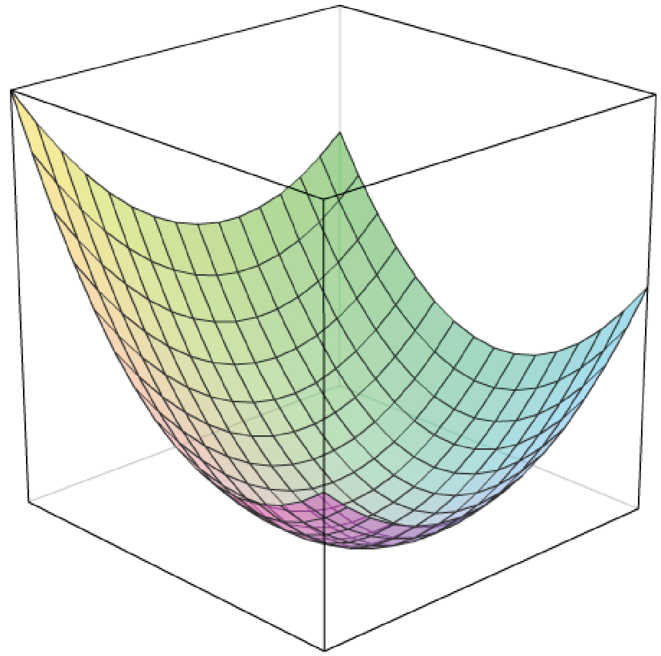
Multivariable Functions (continued)

➲ Example ❶

Consider: $f(x,y) = 3x^2 - xy + 2y^2 - 4x - 7y + 12$

FOCs:

- $f_x = 6x - y - 4 = 0$
- $f_y = -x + 4y - 7 = 0$
- One critical point at: $(x^*, y^*) = (1,2)$



SOC:

- $f_{xx} = 6$
- $f_{yy} = 4$
- $f_{xy} = f_{yx} = -1$

$\rightarrow f_{xx} \times f_{yy} = 24 > \left(f_{xy}\right)^2 = (-1)^2 = 1$

- $f(x,y) = 3x^2 - xy + 2y^2 - 4x - 7y + 12$ has extremum at $(x^*, y^*) = (1,2)$ which is a minimum.

Multivariable Functions (continued)

➲ Example ❷

Consider: $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

FOCs:

- $f_x = 6xy - 6x = 0$
  - $f_x = x(y - 1) = 0$ which is true for $x = 0$, or $y = 1$ or both
- $f_y = 3x^2 + 3y^2 - 6y = 0$
  - Simplified: $x^2 + y^2 - 2y = 0$
  - If $x = 0$:
    - ➢ $y^2 - 2y = y(y - 2) = 0$
    - ➢ Is fulfilled for $y = 0$ or $y = 2$
  - If $y = 1$:
    - ➢ $x^2 + 1 - 2 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm\sqrt{1} = \pm 1$
    - ➢ Is fulfilled for $x = 1$ and $x = -1$
- We have four critical points: $(x^*, y^*) = (0,0)$, $(x^*, y^*) = (0,2)$, $(x^*, y^*) = (1,1)$ and $(x^*, y^*) = (-1,1)$

Multivariable Functions (continued)

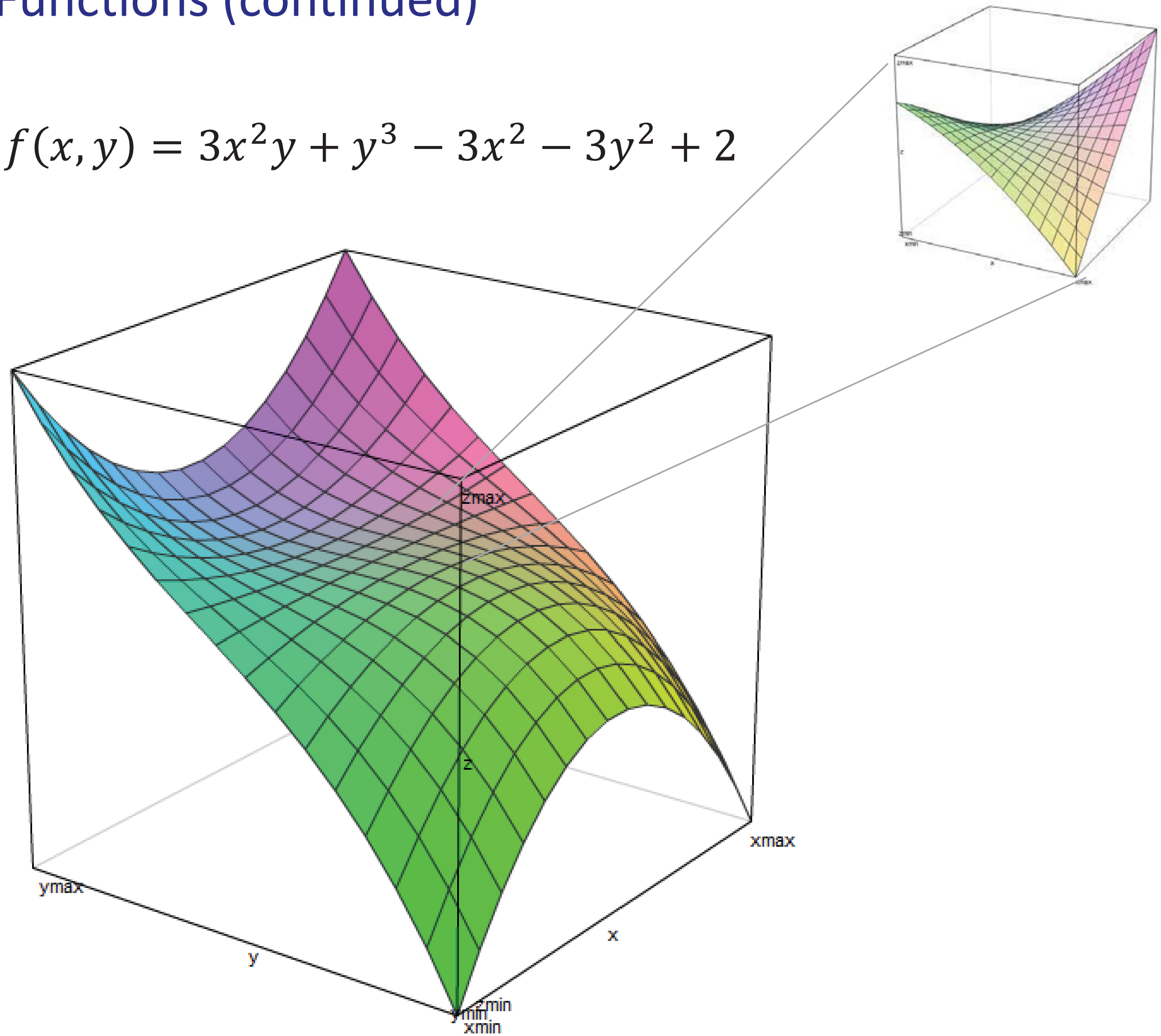
- Example ❷ continued

  SOCs at each of the critical points $(0,0)$, $(0,2)$, $(1,1)$ and $(-1,1)$:

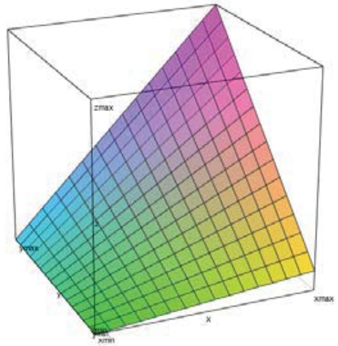
  - o $f_{xx} = 6y - 6$
  - o $f_{yy} = 6y - 6 = f_{xx}$
  - o $f_{xy} = f_{yx} = 6x$
    - At critical point at: $(x^*, y^*) = (0,0)$
      - $f_{xx} = f_{yy} = -6$ and $f_{xy} = 0 \rightarrow f_{xx} \times f_{yy} = 36 > 0 = \left(f_{xy}\right)^2$
      - The function has a maximum at $(0,0)$
    - At critical point at: $(x^*, y^*) = (0,2)$
      - $f_{xx} = f_{yy} = 6$ and $f_{xy} = 0 \rightarrow f_{xx} \times f_{yy} = 36 > 0 = \left(f_{xy}\right)^2$
      - The function has a minimum at $(0,2)$
    - At critical point at: $(x^*, y^*) = (-1,1)$
      - $f_{xx} = f_{yy} = 0$ and $f_{xy} = -6 \rightarrow f_{xx} \times f_{yy} = 0 < 36 = \left(f_{xy}\right)^2$
      - The function has an inflection point at $(-1,1)$
    - At critical point at: $(x^*, y^*) = (1,1)$
      - $f_{xx} = f_{yy} = 0$ and $f_{xy} = 6 \rightarrow f_{xx} \times f_{yy} = 0 < 36 = \left(f_{xy}\right)^2$
      - The function has a inflection point at $(1,1)$

Multivariable Functions (continued)

➲ Example ❷: $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$



Constrained Optimization



Simple problem: Utility maximization

objective function
(utility function)

Budget constraint

$$\text{Max/Min } U = xy + 2x \quad \text{subject to (s.t.)} \quad 4x + 2y = 60$$

Note:

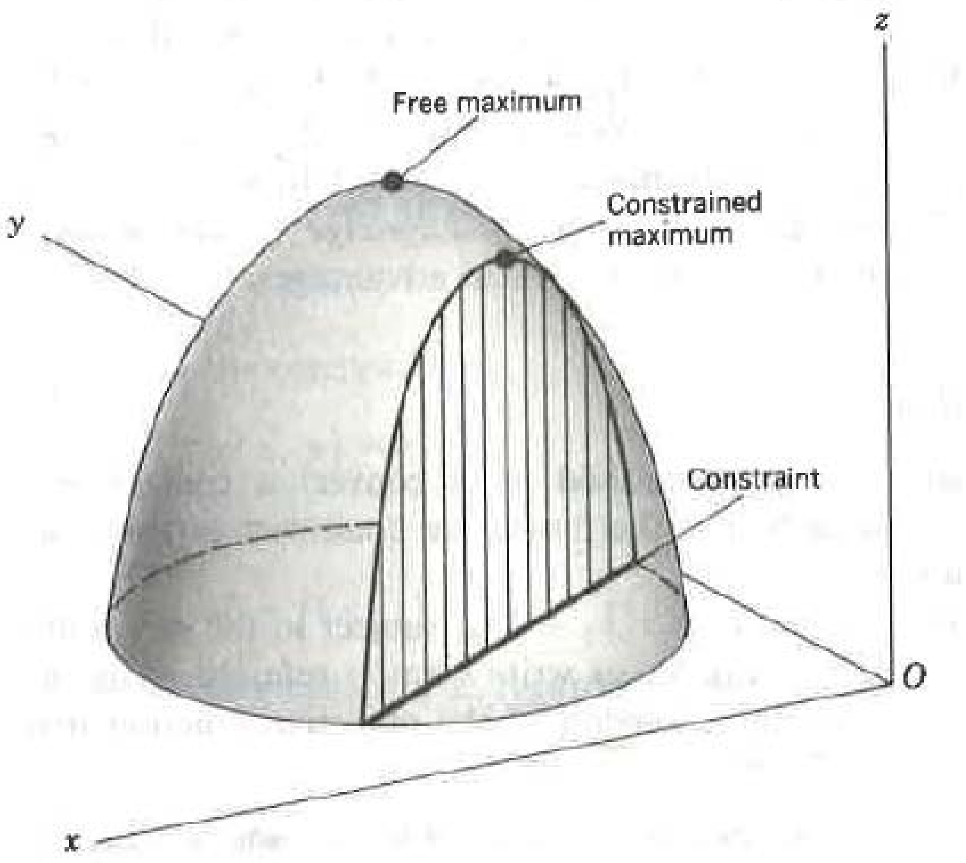
- $\frac{\partial U}{\partial x} > 0$ and $\frac{\partial U}{\partial y} > 0$ (for $x, y \in \mathbb{R}^+$) → to maximize U without constraints, consumer needs to buy an *infinite* amount of both goods
- Budget constraint restricts the optimization problem taking into account the wealth of the consumer

Simple solution:

- Use restriction to find: $y = \frac{60-4x}{2} = 30 - 2x$
- Substitute in $U = x(30 - 2x) + 2x = -2x^2 + 32x$
- Solve single variable problem:
  - FOC: $\frac{\partial U}{\partial x} = -4x + 32 = 0$ → $x = 8$ and $y = 14$
  - SOC: $\frac{\partial^2 U}{\partial x^2} = -4 > 0$: objective function is maximized at $(x^*, y^*) = (8{,}14)$

Constrained Optimization

Illustration of a function with two variables $z = f(x, y)$



Chiang (1984: 371)

Constrained Optimization

The Lagrange Multiplier Method

- Solution to a problem

  Max/Min $f(x_1, x_2)$ subject to (s.t.) $g(x_1, x_2) = c$

- Can be obtained by:

  1. Write down the Lagrangian function:

     $$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

     where λ is the *Lagrange multiplier*.

  2. Differentiate $\mathcal{L}$ wrt x_1, x_2, and λ, and equate all to 0 (**FOCs**):

     $$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 0$$

     $$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 0$$

     $$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = c - g(x_1, x_2) = 0$$

  3. Solve for the three unknowns x_1, x_2, and λ to find critical values

Constrained Optimization

The Lagrange Multiplier Method

➲ Example: max $U = xy + 2x$ s.t. $4x + 2y = 60$

1. Write down the Lagrangian function: $\mathcal{L}(x, y, \lambda) = xy + 2x + \lambda[60 - 4x - 2y]$
2. Differentiate $\mathcal{L}$ wrt x, y, and λ, and equate all to 0 (**FOCs**):

$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial x} = y + 2 - 4\lambda = 0$$

$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial y} = x - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}(x, y, \lambda)}{\partial \lambda} = 60 - 4x - 2y = 0$$

3. Solve for the three unknowns x, y, and λ to find critical values
   - From 1st eq: $\lambda = \frac{y}{4} + \frac{1}{2}$ and from 2nd eq.: $\lambda = \frac{x}{2}$
     - Thus: $\frac{y}{4} + \frac{1}{2} = \frac{x}{2}$ or $x = \frac{y}{2} + 1$
   - In 3rd eq. $4\left(\frac{y}{2} + 1\right) + 2y = 60 \rightarrow y = 14$
   - From 3rd eq. follows: $x = 8$
   - From 1st or 2nd eq. follows: $\lambda = 4$
   - Critical values: $(x^*, y^*, \lambda^*) = (8,14,4)$

Constrained Optimization (continued)

Significance of the Lagrange Multiplier

- Lagrange multiplier λ *approximates* the marginal impact on objective function caused by a small change in the constraint
  - A 1-unit increase (decrease) in c would cause optimal value of $f(x_1, x_2)$ to increase (decrease) by approximately λ-units
    - Last example: U_{max} should increase by 4 units if constraint increases from $4x + 2y = 60$ to $4x + 2y = 61$
    - Economic interpretation: When the budget constraint changes by 1 CHF, utility changes by 4 units = the **marginal utility** of 1 CHF
  - λ referred to as **shadow price** of resource

- Note: $\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$ is equivalent to $\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[g(x_1, x_2) - c]$
  - Critical values are identical
  - Sign of λ differs

Constrained Optimization (contd)
Lagrange multiplier method

Example 2:

Optimise $f(x, y) = 4x^2 + 3xy + 6y^2$ s.t. $g(x, y) = x + y = 56$

Set up Lagrangian:
$\mathcal{L}(x, y, z) = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$

Differentiate wrt x, y and λ and set equal to zero:

$$\frac{\partial \mathcal{L}}{\partial x} = 8x + 3y - \lambda = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 3\mathrm{x} + 12y - \lambda = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 56 - x - y = 0 \qquad (3)$$

Solve for x, y and λ:

- Subtract (2) from (1) to get $5x - 9y = 0 \Rightarrow x = \frac{9}{5y}$
- Substitute into (3): $56 - \frac{9}{5y} - y = 0 \Rightarrow y = 20, \Rightarrow x = 36 \Rightarrow \lambda = 348.$

Constrained Optimization (contd)
Lagrange multiplier method

Critical values $(x^*, y^*, \lambda^*) = (36{,}20{,}348)$

Is this a minimum or a maximum?

Bordered Hessian determinant:

$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| > 0 \Rightarrow$ Local maximum

$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| < 0 \Rightarrow$ Local minimum

$$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix}$$

$$= 0 - (-g_x)\left(-g_x.\mathcal{L}_{yy} - \mathcal{L}_{xy}.-g_y\right) + \left(-g_y\right)\left(-g_x.\mathcal{L}_{yx} - \mathcal{L}_{xx}.-g_y\right)$$

Constrained Optimization (contd)
Lagrange multiplier method

Second order partials: Direct: $\mathcal{L}_{xx} = 8$, $\mathcal{L}_{yy} = 12$; Cross: $\mathcal{L}_{xy} = 3 = \mathcal{L}_{yx}$

$g(x, y) = x + y \Rightarrow$ $g_x = 1$; $g_y = 1$

Calculate the determinant of the bordered Hessian:

$$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 8 & 3 \\ -1 & 3 & 12 \end{vmatrix} = -14 < 0$$

Critical values $(x^*, y^*, \lambda^*) = (36{,}20{,}348)$ is a local minimum

Constrained Optimization (contd)
Lagrange multiplier method

Back to our first example:

Compute the local max/min of $f(x, y) = \ xy + 2x \ s.t. \ g(x, y) = 4x + 2y = 60$

$$\mathcal{L}(x, y, z) = xy + 2\text{x} + \lambda(60 - 4x - 2y)$$

- Critical value: $(x^*, y^*, \lambda^*) = (8{,}14{,}4)$

Maximum or minimum?

Constrained Optimization (contd)
Lagrange multiplier method

First order partials:

$$\frac{\partial \mathcal{L}}{\partial x} = \mathcal{L}_x = y + 2 - 4\lambda \qquad \frac{\partial \mathcal{L}}{\partial y} = \mathcal{L}_y = x - 2\lambda$$

Second order partials:

Direct: $\mathcal{L}_{xx} = 0$; $\mathcal{L}_{yy} = 0$

Cross: $\mathcal{L}_{xy} = 1 = \mathcal{L}_{yx}$

$g(x, y) = 4x + 2y = 60 \Rightarrow g_x = 4$, $g_y = 2$.

Calculate the determinant of the bordered Hessian:

$$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = 16 > 0$$

$(x^*, y^*, \lambda^*) = (8,14,4)$ is a local maximum

Constrained Optimization (continued)

Multiconstraint Case

- Consider the following problem with n variables and m constraints:

$$Max/Min \quad f(x_1, \dots, x_n) \; s.t. \begin{cases} g^1(x_1, \dots, x_n) = c_1 \\ \vdots \\ g^m(x_1, \dots, x_n) = c_m \end{cases}$$

- The Lagrangian is:

$$\mathcal{L}(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(x_1, \dots, x_n) + \sum_{j=1}^{m} \lambda_j \left[c_j - g^j(x_1, \dots, x_n)\right]$$

- FOCs write:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f(x_1, \dots, x_n)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g^j(x_1, \dots, x_n)}{\partial x_i} = 0, \qquad i = 1, \dots, n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = c_j - g^j(x_1, \dots, x_n) = 0, \qquad j = 1, \dots, m$$

Constrained Optimization (continued)

Example

Optimize $f(x, y, z) = x^2 + y^2 + z^2 \quad s.t. \begin{cases} g^1(x, y, z) = x + y + z = 1 \\ g^2(x, y, z) = x + 2y + 3z = 6 \end{cases}$

- Step 1: Set up the Lagrangian

  $$\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 + \lambda_1[1 - x - y - z] + \lambda_2[6 - x - 2y - 3z]$$

- Step 2: Take the first partial derivatives wrt x, y, z, λ_1, and λ_2:

  $$\frac{\partial\mathcal{L}(x, y, z)}{\partial x} = 2x - \lambda_1 - \lambda_2 = 0 \qquad \frac{\partial\mathcal{L}(x, y, z)}{\partial \lambda_1} = 1 - x - y - z = 0$$

  $$\frac{\partial\mathcal{L}(x, y, z)}{\partial y} = 2y - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial\mathcal{L}(x, y, z)}{\partial \lambda_1} = 6 - x - 2y - 3z = 0$$

  $$\frac{\partial\mathcal{L}(x, y, z)}{\partial z} = 2z - \lambda_1 - 3\lambda_2 = 0$$

➲ We know the critical values: $(x^*, y^*, z^*) = \left(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3}\right)$ for $(\lambda_1^*, \lambda_2^*) = \left(-\frac{22}{3}, 4\right)$