Refresher Course in Calculus, Probability, and Statistics

Day 5b: Matrix Operations

Linear Algebra

- Matrix: rectangular array (= collection) of numbers
- A $m \times n$ (read: m-by-n) matrix has m rows and n columns

$$\underset{m\times n}{A} = \left(a_{ij}\right)_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\,j-1} & a_{1\,j} & a_{1\,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,j-1} & a_{2\,j} & a_{2\,j-1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj-1} & a_{mj} & a_{mj+1} & \cdots & a_{mn} \end{pmatrix}$$

where $\left(a_{ij}\right)_{m\times n}$ denotes element a in row i and column j

If:

- m=1 and n>1: row vector
- m>1 and n=1: column vector
- m=n: square matrix
- m=n=1: scalar

James Joseph Sylvester,
1814-1897

Matrix operations: equality of matrices

- Let
  - $\underset{m\times n}{\boldsymbol{\mathcal{A}}} = (a_{ij})_{m\times n}$
  - $\underset{p\times q}{\boldsymbol{\mathcal{B}}} = (b_{ij})_{p\times q}$
  - $\underset{1\times 1}{\alpha}$

- $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are **equal** iif:
  - Same order (same dimensions): $m = p$ <u>and</u> $n = q$
  - Identical elements in corresponding locations: $a_{ij} = b_{ij}$ for all i and j

- Examples:

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} \neq \begin{pmatrix} 5 & 0 \\ 4 & 2 \end{pmatrix} \neq \begin{pmatrix} 4 & 2 & 1 \\ 5 & 0 & -5 \end{pmatrix}$$

Matrix operations: addition and substraction

- Matrices can only be added or substracted if they are of the same order (have the same dimensions): **Conformability condition for addition**
- Matrix *addition*: sum each pair of corresponding elements

$$\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}} = (a_{ij}) + (b_{ij})$$

- Example:

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+2 \\ 5+2 & 0+(-3) \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 7 & -3 \end{pmatrix}$$

- *Substraction* similar:

$$\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}} = (a_{ij}) - (b_{ij})$$

- Example:

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 4-1 & 2-2 \\ 5-2 & 0-(-3) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$$

Matrix operations: Scalar multiplication

- Multiplication by a scalar α: multiply each element by α

$$\alpha \underset{m\times n}{\mathcal{A}} = \left(\alpha \times a_{ij}\right)_{m\times n}$$

  - You can inverse scalar multiplication by extracting common scalars.

- Examples:
  - $3\begin{pmatrix}4 & 2\\ 5 & 0\end{pmatrix} = \begin{pmatrix}12 & 6\\ 15 & 0\end{pmatrix}$
  - $\begin{pmatrix}-2 & -1\\ -\frac{5}{2} & 0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}4 & 2\\ 5 & 0\end{pmatrix}$

Matrix operations: multiplication

- Matrices $\mathcal{A}$ and $\mathcal{B}$ can only be multiplied if the column dimension of $\mathcal{A}$ is equal to the row dimension of $\mathcal{B}$: **Conformability condition for multiplication**
  - with $\underset{m\times n}{\mathcal{A}}$ and $\underset{p\times q}{\mathcal{B}}$, $\mathcal{AB}$ is defined if $n = p$
  - the product matrix $\mathcal{C} = \mathcal{AB}$ has the same number of rows as $\mathcal{A}$ and the same number of columns as $\mathcal{B}$:

$$\underset{m\times n}{\mathcal{A}} \times \underset{p\times q}{\mathcal{B}} = \underset{m\times q}{\mathcal{C}}$$

- where c_{ij} is computed from the elements in the i^{th} row of $\mathcal{A}$ and the those in the j^{th} column of $\mathcal{B}$ as follows:
  1) Pair all the elements sequentially
  2) Multiply out each pair
  3) Sum the resulting products

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Note: there is no Matrix division

Matrix operations: multiplication (continued)

- Example 1

$$\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{A}} \\ 2\times 3}} \times \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 2 \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{B}} \\ 3\times 2}} = \begin{pmatrix} 1\cdot 1 + 0\cdot 2 + 3\cdot 6 & 1\cdot 3 + 0\cdot 5 + 3\cdot 2 \\ 2\cdot 1 + 1\cdot 2 + 5\cdot 6 & 2\cdot 3 + 1\cdot 5 + 5\cdot 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 19 & 9 \\ 34 & 21 \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{C}} \\ 2\times 2}}$$

- Example 2

$$\underbrace{\begin{pmatrix} 1 & 3 \\ -2 & 8 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{A}} \\ 3\times 2}} \times \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{B}} \\ 2\times 1}} = \begin{pmatrix} 1\cdot 2 + 3\cdot 1 \\ -2\cdot 2 + 8\cdot 1 \\ 0\cdot 2 + \frac{1}{2}\cdot 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 4 \\ \frac{1}{2} \end{pmatrix}}_{\substack{\boldsymbol{\mathcal{C}} \\ 3\times 1}}$$

Matrix operations: properties

- *Matrix addition*
  - Communitative: $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$
  - Associative: $(\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$
- *Matrix multiplication*
  - Not communitative: $\mathcal{AB} \neq \mathcal{BA}$ (but with exceptions)
  - Associative: $(\mathcal{AB})\mathcal{C} = \mathcal{A}(\mathcal{BC})$
  - Distributive
    - Pre-multiplication with $\mathcal{A}$: $\mathcal{A}(\mathcal{B} + \mathcal{C}) = \mathcal{AB} + \mathcal{AC}$
    - Post-multiplication with $\mathcal{A}$: $(\mathcal{B} + \mathcal{C})\mathcal{A} = \mathcal{BA} + \mathcal{CA}$
- *Idiosyncracies of matrix algebra*
  - There is no matrix division
  - $\mathcal{AB} = 0$ does not mean that $\mathcal{A} = 0$ or $\mathcal{B} = 0$

    → Happens when $\mathcal{A}$ and $\mathcal{B}$ are singular
  - $\mathcal{AB} = \mathcal{AC}$ and $\mathcal{A} \neq 0$ does not imply $\mathcal{B} = \mathcal{C}$

    → Can happen when $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are singular

Matrix operations: powers

- For square matrix $\underset{n\times n}{\mathcal{A}}$:

$$\begin{aligned} \mathcal{A}\mathcal{A} &= \mathcal{A}^2 \\ \mathcal{A}\mathcal{A}\mathcal{A} &= \mathcal{A}^3 \\ \vdots \quad & \quad \vdots \\ \underset{1}{\mathcal{A}}\underset{2}{\mathcal{A}}\cdots\underset{N}{\mathcal{A}} &= \mathcal{A}^N \end{aligned}$$

Example: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 4 & 1\cdot 2 + 2\cdot 3 \\ 4\cdot 1 + 3\cdot 4 & 4\cdot 2 + 3\cdot 3 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$

- **Idempotent matrix**: $\mathcal{A}^2 = \mathcal{A}$

Example: $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + 1\cdot(-2) & 2\cdot 1 + 1\cdot(-1) \\ -2\cdot 2 + (-1)(-2) & -2\cdot 1 + (-1)(-1) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$

- **Diagonal matrix**: $\mathcal{D}^m = \left(\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}\right)^m = \begin{bmatrix} a_{11}^m & 0 & \cdots & 0 \\ 0 & a_{22}^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^m \end{bmatrix}$

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 0 & 1\cdot 0 + 0\cdot 2 \\ 0\cdot 1 + 2\cdot 0 & 0\cdot 0 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Identity matrix

- Idenitity matrix: **square** matrix with *ones* along its principal diagonal and *zeros* elsewhere:

$$\mathcal{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Same role as 1 in scalar algebra:
  - $\underset{m\times n}{\mathcal{A}}\,\mathcal{I}_n = \mathcal{I}_m \underset{m\times n}{\mathcal{A}} = \underset{m\times n}{\mathcal{A}}$
    - You can insert or delete an identity matrix without affecting a matrix product: $\underset{m\times n}{\mathcal{A}}\;\underset{n\times n}{\mathcal{I}}\;\underset{n\times p}{\mathcal{B}} = \underset{m\times n}{\mathcal{A}}\;\underset{n\times p}{\mathcal{B}}$
  - $(\mathcal{I}_n)^k = \mathcal{I}_n$ (idempotent matrix)

Null matrix

➲ Null matrix (or zero matrix): matrix with zeros everywhere

$$\underset{2\times 2}{\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \underset{2\times 3}{\mathbf{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Same role as 0 in scalar algebra:

$$\underset{m\times n}{\boldsymbol{\mathcal{A}}} + \underset{m\times n}{\mathbf{0}} = \underset{m\times n}{\mathbf{0}} + \underset{m\times n}{\boldsymbol{\mathcal{A}}} = \underset{m\times n}{\boldsymbol{\mathcal{A}}}$$

$$\underset{m\times n}{\boldsymbol{\mathcal{A}}} \times \underset{n\times p}{\mathbf{0}} = \underset{m\times p}{\mathbf{0}} \text{ and } \underset{q\times m}{\mathbf{0}} \times \underset{m\times n}{\boldsymbol{\mathcal{A}}} = \underset{q\times n}{\mathbf{0}}$$

Matrix operations: transpose

- Interchanging rows and columns of a matrix $\boldsymbol{\mathcal{A}}$ yields its **transpose**, denoted $\boldsymbol{\mathcal{A}}'$ or $\boldsymbol{\mathcal{A}}^T$
  - Example:

$$\underset{2\times 3}{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} 3 & 8 & -2 \\ 0 & 1 & 9 \end{pmatrix} \qquad \underset{3\times 2}{\boldsymbol{\mathcal{A}}}' = \begin{pmatrix} 3 & 0 \\ 8 & 1 \\ -2 & 9 \end{pmatrix}$$

  - This is equivalent to reflecting the elements along the matrix main diagonal.

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Source: wikipedia

- Properties of transposes:
  - $\underset{m\times n}{\boldsymbol{\mathcal{A}}} \rightarrow \underset{n\times m}{\boldsymbol{\mathcal{A}}}'$
  - $(\boldsymbol{\mathcal{A}}')' = \boldsymbol{\mathcal{A}}$ (transpose is an involution)
  - $(\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}})' = \boldsymbol{\mathcal{A}}' + \boldsymbol{\mathcal{B}}'$ (transpose respects addition)
  - $(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}})' = \boldsymbol{\mathcal{B}}'\boldsymbol{\mathcal{A}}'$ (the transpose of a product is the product of transposes in *reverse* order)
  - $(\alpha\boldsymbol{\mathcal{A}})' = \alpha\boldsymbol{\mathcal{A}}'$ (the transpose of a scalar is the same scalar)
  - If $\underset{n\times n}{\boldsymbol{\mathcal{A}}}' = \underset{n\times n}{\boldsymbol{\mathcal{A}}}$ (**symmetric matrix**, along its main diagonal)

Example: $\mathcal{D} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -4 & 3 \\ -1 & 3 & 9 \end{pmatrix} = \mathcal{D}' = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -4 & 3 \\ -1 & 3 & 9 \end{pmatrix}$

Matrix operations: determinant

- Determinant of a **square** matrix $\boldsymbol{\mathcal{A}}$, denoted $|\boldsymbol{\mathcal{A}}|$ or $det(\boldsymbol{\mathcal{A}})$: scalar derived from $\boldsymbol{\mathcal{A}}$
- 2 x 2 Matrix:

$$\underset{2\times 2}{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|\boldsymbol{\mathcal{A}}| \ = a_{11}a_{22} - a_{12}a_{21}$$

  - Difference between the product of the diagonal elements and the product of the off-diagonal elements
  - The determinant is of order 2.
- Example:

$$\underset{2\times 2}{\boldsymbol{\mathcal{A}}} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$|\boldsymbol{\mathcal{A}}| \ = 3 \cdot 1 - 2 \cdot 1 = 1$$

Matrix operations: Inverse

- Inverse of a **square** matrix $\boldsymbol{\mathcal{A}}$, denoted $\boldsymbol{\mathcal{A}}^{-1}$ is defined by

$$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}^{-\mathbf{1}} = \boldsymbol{\mathcal{A}}^{-1}\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{I}}$$

  → like in scalar algebra, where $\alpha\alpha^{-1} = 1$ (if $\alpha \neq 0$)

- A matrix will have an inverse (i.e. it is *invertible*) iif:
  - it‘s of the order $n \times n$ (= it‘s square): *necessary* condition
  - it's nonsingular, i.e. $|\boldsymbol{\mathcal{A}}| \neq 0$: *sufficient* condition

- Properties of inverses:
  - $(\boldsymbol{\mathcal{A}}^{-1})^{-1} = \boldsymbol{\mathcal{A}}$ (inverse is an involution)
  - $(\boldsymbol{\mathcal{A}}^{-1})' = (\boldsymbol{\mathcal{A}}')^{-\mathbf{1}}$ (inverse of the transpose is the transpose of the inverse)
  - $(\alpha\boldsymbol{\mathcal{A}})^{-1} = \alpha^{-1}\boldsymbol{\mathcal{A}}^{-1}$ for $\alpha \neq 0$

➲ 2 x 2 Matrix:

$$\underset{2\times 2}{\mathcal{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|\mathcal{A}| = a_{11}a_{22} - a_{12}a_{21}$$

Inverse matrix:

$$\mathcal{A}^{-1} = \frac{1}{|\mathcal{A}|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Extra slides on the computation of inverse matrix

Matrix operations: determinant (*)

- n x n Matrix
- Evaluation of an n_{th} order matrix by Laplace extension (or: cofactor extension):
  - Determinant of $\underset{n \times n}{\boldsymbol{\mathcal{A}}}$: weighted sum of the determinants of *n* submatrices of $\boldsymbol{\mathcal{A}}$, each of size $(n-1) \times (n-1)$
  - **Minor** of element a_{ij}, denoted M_{ij}: subdeterminant of $\boldsymbol{\mathcal{A}}$ obtained by deleting the row i and column j of $|\boldsymbol{\mathcal{A}}|$
  - **Cofactor**, denoted C_{ij}: minor M_{ij} with a sign attached to it:
    - $C_{ij} = (-1)^{i+j} M_{ij}$
  - Determinant of $\boldsymbol{\mathcal{A}}$:

$$|\boldsymbol{\mathcal{A}}| = \sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} a_{ij} C_{ij} \quad \text{Expansion by } i^{th} \text{ row}$$

$$= \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} a_{ij} C_{ij} \quad \text{Expansion by } j^{th} \text{ column}$$

Pierre-Simon Laplace

Matrix operations: Inverse

Computing the inverse:

1. Using the general formula

$$\boldsymbol{\mathcal{A}}^{-1} = \frac{1}{|\boldsymbol{\mathcal{A}}|} \cdot adj(\boldsymbol{\mathcal{A}})$$

where $adj(\boldsymbol{\mathcal{A}})$ (read: adjoint/adjugate of $\boldsymbol{\mathcal{A}}$) is the transpose of the cofactor matrix $\boldsymbol{\mathcal{C}}^{+} = \left(\mathcal{C}_{ij}\right)$

$$adj(\boldsymbol{\mathcal{A}}) = (\boldsymbol{\mathcal{C}}^{+})^{\boldsymbol{T}} = \begin{pmatrix} \mathcal{C}_{\mathbf{11}} & \mathcal{C}_{\mathbf{21}} & \cdots & \mathcal{C}_{\boldsymbol{n}\mathbf{1}} \\ \mathcal{C}_{\mathbf{12}} & \mathcal{C}_{\mathbf{22}} & \cdots & \mathcal{C}_{\boldsymbol{n}\mathbf{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{\mathbf{1}\boldsymbol{n}} & \mathcal{C}_{\mathbf{2}\boldsymbol{n}} & \cdots & \mathcal{C}_{\boldsymbol{nn}} \end{pmatrix}$$

Matrix operations: Inverse (continued)

Computing the inverse using the general formula (continued):

- Example :

$$\mathcal{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

- General form of the inverse: $\mathcal{A}^{-1} = \frac{1}{|\mathcal{A}|} \cdot adj(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \cdot (\mathcal{C}^{+})^{T}$
- Determinant: $|\mathcal{A}| = 1\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$
- Co-factor matrix: $\mathcal{C}^{+} =$

$$\begin{bmatrix} \mathcal{C}_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7 & \mathcal{C}_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1 & \mathcal{C}_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1 \\ \mathcal{C}_{21} = -\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 3 & \mathcal{C}_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0 & \mathcal{C}_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1 \\ \mathcal{C}_{31} = \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3 & \mathcal{C}_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1 & \mathcal{C}_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}$$

Matrix operations: Inverse (continued)

- Transpose of the Co-factor matrix

$$(\boldsymbol{\mathcal{C}}^{+})^{T} = \begin{bmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

- Hence the inverse of $\boldsymbol{\mathcal{A}}$,

$$\boldsymbol{\mathcal{A}}^{-\mathbf{1}} = \frac{1}{|\boldsymbol{\mathcal{A}}|}(\boldsymbol{\mathcal{C}}^{+})^{T} = \frac{1}{-1}\begin{pmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$